

Searching for the Gaussian process of point processes

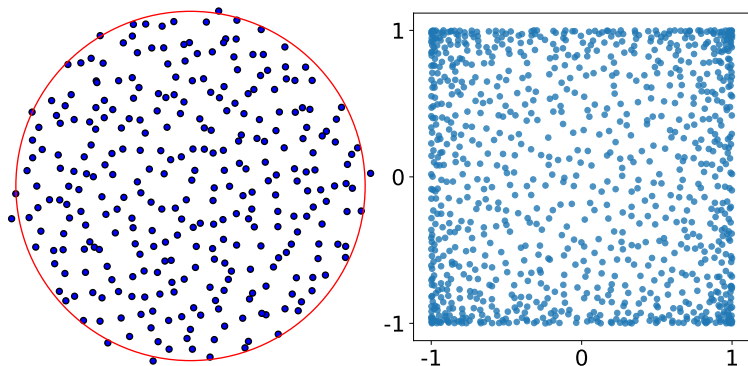
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Two DPP samples



Check out our [Python toolbox DPPy](#)¹.

¹Gautier, Bardenet, Polito, and Valko, 2019, github.com/guilgautier/DPPy.

Integration with DPPs

Integration with Coulomb gases

- ▶ Consider the RKHS \mathcal{F} with kernel k , i.e. the completion of

$$\left\{ \sum_{i=1}^M \alpha_i k(x_i, \cdot), M \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{R}, x_1, \dots, x_M \in \mathbb{R}^d \right\}.$$

for the inner product defined by $\langle k(x, \cdot), k(y, \cdot) \rangle_{\mathcal{F}} := k(x, y)$.

- ▶ As an example, consider the kernel on $[0, 1]$ given by

$$k_s(x, y) = 1 + 2 \sum_{m \in \mathbb{N}^*} \frac{1}{m^{2s}} \cos(2\pi m(x - y)),$$

where the convergence holds uniformly on $[0, 1] \times [0, 1]$. The RKHS $\mathcal{F} = \mathcal{F}_s$ is the **periodic Sobolev space** of smoothness parameter s .

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Example: periodic Sobolev spaces

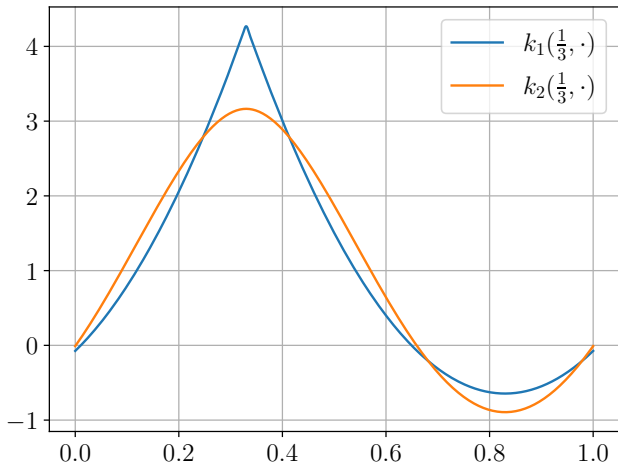


Figure: The Sobolev kernel, with smoothness parameter $s \in \{1, 2\}$.

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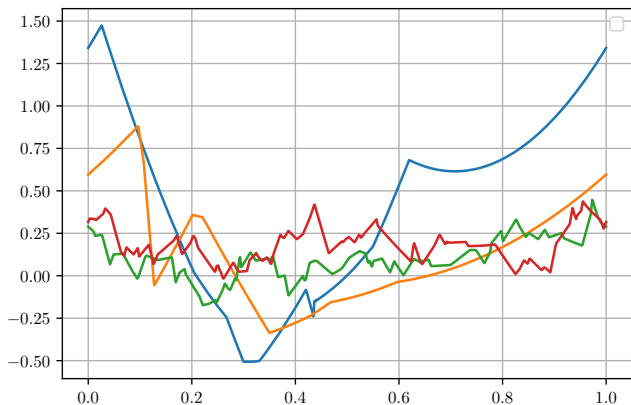


Figure: A few functions of unit norm in \mathcal{F}_s , with $s = 1$.

Example: periodic Sobolev spaces

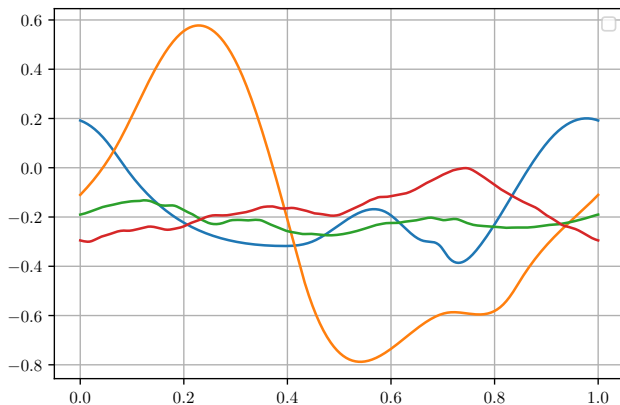


Figure: A few functions of unit norm in \mathcal{F}_s , with $s = 2$.

- ▶ Under general assumptions,

$$\Sigma : f \mapsto \int k(x, \cdot) f(x) d\mu(x)$$

is a compact self-adjoint operator on $L^2(\mu)$.

- ▶ With slightly more assumptions, there is an orthonormal basis (e_n) of $L^2(\mu) \cap C^0$ and $\sigma_n \rightarrow 0$ such that,

$$k(x, y) = \sum_{n \geq 1} \sigma_n e_n(x) e_n(y),$$

where, for any x , the convergence is uniform in y .

- ▶ In that case, $f \in \mathcal{F}$ if and only if $\sum_n \sigma_n^{-1} |\langle f, e_n \rangle|^2$ converges. In other words,

$$\mathcal{F} \approx \Sigma^{1/2} L^2(\mu).$$

- ▶ Let $f \in \mathcal{F}$, $g \in L^2(d\mu)$ then

$$\left| \int fg d\mu - \sum_{i=1}^N w_i f(x_i) \right| \leq \|f\|_{\mathcal{F}} \left\| \mu_g - \sum_{i=1}^N w_i k(x_i, \cdot) \right\|_{\mathcal{F}}, \quad (1)$$

where

$$\mu_g = \int g(x) k(x, \cdot) d\mu(x)$$

is the **mean element of g** .

- ▶ Once the nodes x_1, \dots, x_N are known, minimizing the RHS of (1) in w boils down to inverting an $N \times N$ matrix.

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Remember $k(x, y) = \sum_{n \geq 1} \sigma_n e_n(x) e_n(y)$.

Algorithm 1: DPP

- ▶ Take $K(x, y) = \sum_{n=1}^N e_n(x) e_n(y)$.
- ▶ Let $x_1, \dots, x_N \sim 1/N! \det[K(x_i, x_j)] d\mu(x_1) \dots d\mu(x_N)$.
- ▶ Solve the linear problem for the weights w_1, \dots, w_N .

Theorem (Belhadji, Bardenet, and Chainais, 2019)

Assume $\sum_{n=1}^N |\langle g, e_n \rangle|^2 \leq 1$. Let $r_N = \sum_{m \geq N+1} \sigma_m$, then

$$\mathbb{E} \left\| \mu_g - \sum_{i=1}^N w_i k(x_i, \cdot) \right\|_{\mathcal{F}}^2 \leq 2\sigma_{N+1} + 2 \left(Nr_N + \sum_{\ell=2}^N \frac{\sigma_1}{\ell!^2} \left(\frac{Nr_N}{\sigma_1} \right)^\ell \right).$$

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Volume sampling

- ▶ Let $x_1, \dots, x_N \sim Z^{-1} \det[k(x_i, x_j)] d\mu(x_1) \dots d\mu(x_N)$
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Under a mild condition on (σ_n) ,

$$\mathbb{E} \sup_{\|f\|_{\mathcal{F}} \leq 1} \left| \int f d\mu - \sum_{i=1}^N w_i k(x_i, \cdot) \right|^2 = \mathcal{O}(\sigma_N).$$

- ▶ Pinkus, 2012 shows that $\inf_{\substack{Y \subset \mathcal{F} \\ \dim Y = N}} \sup_{\|g\|_{L^2(\mu)} \leq 1} \inf_{y \in Y} \|\mu_g - y\|_{\mathcal{F}}^2 = \sigma_{N+1}$.

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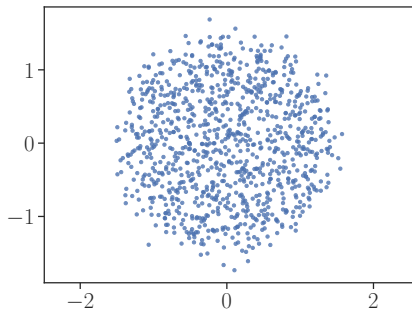
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A new distribution for quadrature nodes

Under conditions on k and V , consider

$$x_1, \dots, x_N \sim \frac{1}{Z_{N, \beta_N}^V} e^{-\frac{\beta_N}{2N^2} \sum_{i \neq j} k(x_i, x_j) - \frac{\beta_N}{N} \sum_{i=1}^N V(x_i)} dx_1 \dots dx_N,$$

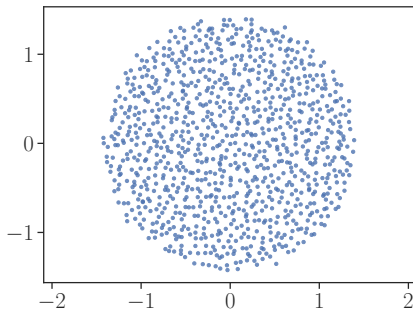


(a) $\beta_N = N^{3/2}$

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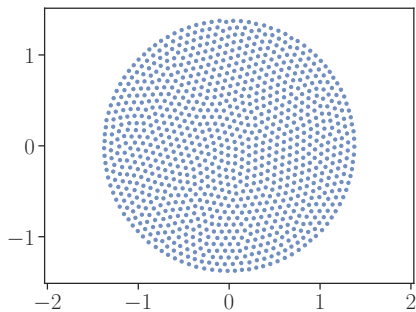


(b) $\beta_N = N^2$

A new distribution for quadrature nodes

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(c) $\beta_N = N^3$

Theorem (Rouault, Bardenet, and Maida, 2024)

Let $\beta_N \gg N$. There exists $u_0, u_1 > 0$ such that for $r \geq u_0/\sqrt{\beta_N}$, the event

$$\sup_{\|f\|_{\mathcal{F}} \leq 1} \left| \int f d\mu_N - \int f d\mu_V \right|^2 \geq r. \quad (2)$$

has probability smaller than $\exp(-u_1 \beta_N r^2)$.

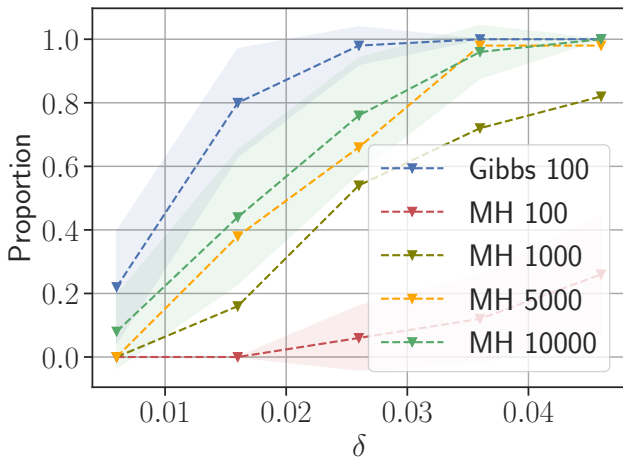
But we want to integrate w.r.t. a target π

- ▶ Assume π has compact support $S \subset B(0, R)$, and has finite entropy.
- ▶ We have $\mu_V = \pi$ if we take e.g.

$$V : y \mapsto - \int k(x, y) d\pi(x) + (\|y\|^2 - R^2) \mathbf{1}_{y \notin B(0, R)}.$$

- ▶ If we approximate V by an MCMC chain of length n , the theorem remains valid! (Rouault, Bardenet, and Maida, 2025).








An experiment to validate coverage



To go further:

- ▶ On PPs and quantum optics (Bardenet et al., 2022).
- ▶ On PPs and DPPs (Hough, Krishnapur, Peres, and Virág, 2006).

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